10. SOLVING POLYNOMIAL EQUATIONS

§10.1. Quadratic Equations

A **quadratic equation** is one of the form

$$ax^2 + bx + c = 0.$$

The coefficients can come from any field, such as the field of real numbers or the field of comples numbers. Whenever quadratics are taught in high-school a lot of effort is expended on teaching students how to factorise quadratics.

Example 1: Factorise $x^2 - 5x + 6$ and hence solve the quadratic $x^2 - 5x + 6 = 0$.
Solution: Suppose the zeros are α and β.
Then $x^2 - 5x + 6 = (x - \alpha)(x - \beta) = x^2 - (\alpha + \beta)x + \alpha\beta$ and so $\alpha + \beta = -5$ and $\alpha\beta = 6$.
What two numbers add to 5 and multiply to 6. Clearly 2 and 3. So $\alpha = 2$ and $\beta = 3$, or the other way round – it doesn't matter. So $x^2 - 5x + 6 = (x - 2)(x - 3)$.

The problem arises when the coefficients have lots of divisors.

Example 2: Solve the quadratic $18x^2 - 33x - 175 = 0$.
Solution: Here we must factorise the quadratic as

$$(ax - b)(cx - d).$$

So $ac = 18$, $bd = 175$ and $ad + bc = 33$. There are just too amy possibilities to try: The possibilities for a are ±1, ±2, ±3, ±6, ±9 and ±18 with corresponding values of c being ±18, ±9, ±6, ±3, ±2 and ±1 respectively. These have to be combined with the possible bd's. Perhaps $b = 1$ and $d = 175$, or $b = –5$ and $d = – 35$, or …

Except in very simple cases, this is a ridiculous technique. What's worse is that the values of a, b might not be rational numbers.

Example 3: Solve the quadratic $x^2 + x – 1 = 0$.
Solution: The problem here is not that of having too many factors to try but rather not enough. In fact the coefficients of the factors are not even rational numbers. The solutions are $\frac{-1 \pm \sqrt{5}}{2}$. If you add these you get – 1, that is minus the coefficient of x and if you multiply them you get –1. Would you have ever thought of trying these?

Trying to factorise difficult quadratics has sent many a student into despair. The moral of the story is that, unless the factorisation jumnps out at you, as in Example 1, always use the quadratic formula.

§10.2. Solving Quadratic Equations

Over the field of real numbers, the solutions to the quadratic $ax^2 + \underline{bx} + c = 0$ are $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. Actually this works for all fields, except those where 1 + 1 = 0, because then the denominator will be zero. Don't worry about that for now. We'll look at such fields in the next chapter.

The other proviso is that the square root of $b^2 - 4ac$ must exist, that is that $b^2 - 4ac \geq 0$. When you learnt quadratic equations at high school you'd never heard of complex numbers, and so you concluded that some quadratic equations have no solutions.

The quantity $b^2 - 4ac$ is called the **discriminant** of the quadratic $ax^2 + bx + c$, and it's often denoted by $\boldsymbol{\Delta}$. The name arose because if you limit yourself to the real numbers, it discriminates between those quadratics that have solutions and those that don't. If $\Delta \geq 0$ there are solutions and if $\Delta < 0$ there are none. But now that we know about complex numbers, *all* quadratics have solutions and the discriminant simply tells you whether the solutions are real.

IF $\Delta > 0$ the quadratic has two distinct real solutions. If $\Delta = 0$ the quadratic has one (repeated) real solution. If $\Delta < 0$ the quadratic has two distinct non-real solutions.

Example 4: Solve the quadratic $x^2 + x + 1$ over the real numbers and over the complex numbers.

Solution: $\Delta = 1^2 - 4 = -3 < 0$, so there are no real solutions.

Over the complex numbers, $x = \dfrac{-1 \pm \sqrt{-3}}{2} = \dfrac{-1 \pm \sqrt{3}\,i}{2}$.

The usual method for deriving the quadratic formula is called *completing the square*. If you really want to see how it's done this way you can consult any book on elementary algebra. But I never teach this method because it doesn't show what's really going on. The following method is no more difficult than completing the square =, but it highlights the role that symmetry plays in the solution of polynomials.

Theorem 1: Provided that $1 + 1 \neq 0$ in the field F, the solutions to the equation $ax^2 + bx + c = 0$ are $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Proof: Let the solutions be α and β. Then the sum of and product of these can be expressed very simply in terms of the coefficients:

$$S = \alpha + \beta = -\frac{b}{a}, \quad P = \alpha\beta = \frac{c}{a}$$

Both the sum of the zeros and the product of the zeros are symmetric in terms of α and β. If α and β are swapped they remain unchanged.

These two functions of the roots are called the *elementary symmetric functions* and, as we have seen in the previous chapter, other symmetric functions of the roots can be expressed in terms of them.

Now an expression such as $\alpha - \beta$ is not symmetric. Swapping α and β in fact changes the sign of the expression. However if we square $\alpha - \beta$, this change of sign disappears and we again get the symmetric function:

$$(\alpha - \beta)^2 = \alpha^2 + \beta^2 - 2\alpha\beta = (\alpha + \beta)^2 - 4\alpha\beta = S^2 - 4PS$$

$$= \left(-\frac{b}{a}\right)^2 - 4\left(\frac{c}{a}\right) \quad = \frac{b^2 - 4ac}{a^2}.$$

Hence we can find the values of $\alpha - \beta$ simply by taking square roots, getting $\alpha - \beta = \dfrac{\pm\sqrt{b^2 - 4ac}}{a}$.

Now $\alpha + \beta = -\dfrac{b}{a}$ and so adding these equations and dividing by 2 we get $\alpha = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. ✋☺

We can solve a quadratic equation over the complex field, with non-real coefficients, by using the same quadratic formula. The only slight difficulty is finding the square roots of a non-real complex number. Remember that to find the square roots of a + bi we convert to polar form, take the square root of the modulus and halve the argument. Then we convert back to the x + iy form.

Example 5: Find the square roots of $1 + \sqrt{3}i$.

Solution: $1 + \sqrt{3}i = r(\cos\theta + i \sin\theta$ for some real $r > 0$ and some real θ with $0 \le \theta < 2\pi$.

r is the modulus $= \sqrt{1 + \sqrt{3}^2} = 2$ and $\tan \theta = \dfrac{\sqrt{3}}{1} = \sqrt{3}$.

So $\theta = \pi/3$.

The square roots are $\pm \sqrt{2}[\cos(\pi/6) + i \sin (\pi/6)]$

$= \pm \sqrt{2}[(\sqrt{3}/2) + (1/2)\, i]$

$= \pm [\sqrt{3}/2 + (1/2)i]$.

Example 6: Solve the quadratic $z^2 + (1 + i)z - i = 0$.

Solution: $z = \dfrac{-(1 + i) \pm \sqrt{6i}}{2}$.

Now the square roots of $i = \cos(\pi/2) + i \sin(\pi/2)$

$= \pm (\cos \pi/4 + i \sin \pi/4)$

$= \pm \dfrac{1}{\sqrt{2}} + \dfrac{i}{\sqrt{2}} = \pm \dfrac{1 + i}{\sqrt{2}}$.

So $z = (1 + i)(-1 \pm \sqrt{3}\,)$

$= (\sqrt{3} - 1) + (\sqrt{3} - 1)i$ and $-(\sqrt{3} + 1) - (\sqrt{3} + 1)i$

Example 7: Find the zeros of the quadratic $z^2 + 2iz + i$.

Solution:

$$z = \frac{-2i \pm \sqrt{-4 - 4i}}{2} .$$

and since $-4 - 4i = \sqrt{32}(\cos 5\pi i/4 + i \sin 5\pi i/4)$ we can express its square roots as

$$\pm 2^{5/4}(\cos 5\pi i/8 + i \sin 5\pi i/8).$$

The zeros of the quadratic are thus:

$$2^{5/4}\cos 5\pi i/8 + (2^{5/4}\sin 5\pi i/8 - 1)i \text{ and}$$
$$-2^{5/4}\cos 5\pi i/8 - (2^{5/4}\sin 5\pi i/8 + 1)i$$

§10.3. Solving Cubics

Every cubic equation can be expressed in the form $x^3 + bx^2 + cx + d = 0$ by dividing through by the leading coefficient. Then by substituting $y = x + \frac{b}{3}$ we can eliminate the squared term and write y as a solution to a cubic of the form $y^3 + cy + d = 0$.

For such a cubic the three solutions for y are

$$\sqrt[3]{\frac{-d}{2} + \sqrt{\frac{d^2}{4} + \frac{c^3}{27}}} + \sqrt[3]{\frac{-d}{2} - \sqrt{\frac{d^2}{4} + \frac{c^3}{27}}}\,.$$

There are three cube roots for every non-zero complex number, so this formula appears to give 9 solutions altogether. However the cube roots have to be paired up in a special way, giving only three solutions in all. If you want to see how this is derived you can find a proof in my notes on *Galois Theory*.

Even if the zeros turn out eventually to be real the process involves working with non-real numbers. This is why, despite the misgivings mathematicians had about whether square roots of –1 really exist, they could not ignore the fact that even if they didn't exist they were really useful.

You'll notice that the cubic formula is of the same type as the quadratic one. You start with the coefficients and then carry out certain arithmetic operations to find the solutions. These operations are addition, subtraction, multiplication and division and the extraction of roots. In the case of the cubic we need to take square roots as well as cube roots.

The cubic formula was discovered in 1515. Calculus would not be invented for more than 150 years, so Newton's Method was not an option. (See my notes *Techniques of Calculus* for a discussion of Newton's Method.)

Even when Newton's Method became available the iterative calculations had to be done by hand. These days we use a computer. If you have any knowledge of spreadsheets I'm sure you'll be able to set up a spreadsheet for solving any cubic with real coefficients.

Once you find a real zero, α, you then divide by $x - \alpha$ to get a quadratic. You solve this by the quadratic formula to get the other two zeros, whether they are both real or a conjugate pair of non-real zeros.

Example 8: Solve the cubic $2x^3 - 5x^2 - 5x - 7 = 0$.
Solution: Using Newton's Method we discover that $\alpha = 3.5$ is a solution. Trying various starting values we always converge to $x = 3.5$, which suggests, though doesn't prove, that the other two solutions are non-real.

Factorising we get $(2x - 7)(x^2 + x + 1) = 0$, and we solve $x^2 + x + 1$ by the quadratic formula to obtain the other two solutions.

So the three solutions are: 3.5 and $\dfrac{-1 \pm \sqrt{3}\,i}{2}$.

To 5 decimal places these are:

$$3.5,\ -0.5 + 1.73205i,\ -0.5 - 1.73205i.$$

Now what do we do if we have a cubic where some of the coefficients are non-real? We clearly can't use Newton's Method. In fact such a cubic may not even have a real zero. Moreover there may not be a pair of conjugate zeros.

In such a case we can use the following technique. Exprees the zero as $x + iy$ and substitute into the cubic. Then equate real and imaginary parts. This will give two equations in two real variables. But they won't be linear equations. We might be lucky in being able to eliminate one variable and substitute, giving a real polynomial. But that's not a technique that can be used in most cases.

However there is a numerical technique of my own that's a generalisation of Newton's Method. Like Newton's Method it gives approximations – in this case to any two equations in two real variables (provided the partial derivatives exist).

§10.4. Solving Quartics

There is, in fact, a quartic formula, discovered in 1545, that gives exact solutions to any quartic. It's a formula, like the quadratic formula, where you start with the coefficients and carry out a series of arithmetic operations, involving addition, subtraction, multiplication, division and the extraction of n'th roots. Like the cubic, it can be derived by using symmetrical expressions in the zeros. I can't emphasise too strongly the role symmetry plays in the derivation of such formulae.

It's a very complicated formula and, like the cubic formula, nobody ever uses it. Now we could try Newton's Method. If we find a real zero α by Newton's Method we can divide the quartic by $x - \alpha$ and break it down to a cubic. But what if the quartic has no real zeros, but rather, two conjugate pairs of non-real zeros?

However, whether or not a quartic has any real zeros it must factorise into a product of two quadratics. One or both of these may factorise into linear factors and we may have some real zeros. But at the very worst the quartic factorises into two real quadratics. If we can find the coefficients of these we can use the quadratic formula. For a start we can divide by the leading coefficient, so we can assume that the quartic is monic.

Suppose that:
$x^4 + px^3 + \mathrm{q}x^2 + rx + s = (x^2 + \mathrm{a}x + b)(x^2 + cx + d)$.

Equating corresponding coefficients we have:

$$\left.\begin{aligned} a + c &= p \\ ac + b + d &= q \\ ad + bc &= r \\ bd &= s \end{aligned}\right\}$$

Hence $c = p - a$ and $d = s/b$. Substituting into the third equation we can express a in terms of b and the coefficients of the quartic. Substituting into the second equation we get a polynomial in b whose coefficients can be expressed in terms of those of the quartic.

We can then get the real zeros of this polynomial and, for each, we can find the corresponding values of a, c and d. We then find the zeros of each of the quadratic factors by the quadratic formula and hence obtain the four zeros of the quartic.

Example 9: Solve the equation

$$x^4 + 2x^3 + 4x^2 + 3x + 2 = 0$$

completely over the complex numbers.
Solution: A sketch of the graph of

$$y = x^4 + 2x^3 + 4x^2 + 3x + 2$$

reveals that there are no real zeros.

Write $x^4 + 2x^3 + 4x^2 + 3x + 2 = (x^2 + \mathrm{a}x + \mathrm{b})(x^2 + \mathrm{c}x + \mathrm{d})$ where a, b, c, d are real and where these quadratic factors have no real zeros.

Equating coefficients we get the system of equations:

$$\left.\begin{array}{c} a + c = 2 \\ ac + b + d = 4 \\ ad + bc = 3 \\ bd = 2 \end{array}\right\}$$

Hence $c = 2 - a$ and $d = 2/b$.

Substituting, and simplifying we get the system:

$$\left.\begin{array}{c} 2ab - a^2b + b^2 - 4b + 2 = 0 \\ 2a + 2b^2 - ab^2 - 3b = 0 \end{array}\right\}$$

From the second equation, $a = \dfrac{b(3 - 2b)}{2 - b^2}$

Substituting into the first equation we get:

$$b^6 - 4b^5 + 4b^4 - b^3 + 8b^2 - 16b + 8 = 0.$$

Fortunately we only want real zeros, so we use Newton's Method to show that $b = 1$ and $b = 2$ are the real zeros.

If $b = 1$, $d = 2$, $a = 1$ and $c = 1$.

If $b = 2$, $d = 1$, $a = 1$ and $c = 1$.

So $x^4 + 2x^3 + 4x^2 + 3x + 2 = (x^2 + x + 1)(x^2 + x + 2)$.

Solving these quadratics we obtain the four zeros of the quartic: $x = \dfrac{-1 \pm \sqrt{3}\mathrm{i}}{2}, \dfrac{-1 \pm \sqrt{7}\mathrm{i}}{2}$

§10.5. Solving Quintics And Beyond

A quintic is a polynomial of degree 5. A real quintic has at least one real zero, which we can find by using Newton's Method. Then we can break it down to a quartic and solve it by the above methods. No problem.

Could we use the quintic formula? If there was one it would be exceedingly complicated, but the simple fact is that there is none!

Perhaps some very clever mathematician find one some day? Never! It has been proved that no such formula, along the lines that we've been talking about, can possibly exist. This was proved by the Norwegian mathematiciam Abel.

Some special quintics can be solved by starting with the coefficients and performing calculations using the operations of addition, subtraction, multiplication, division and extraction of n'th roots. Such polynomials are said to be **soluble by radicals**.

Abel proved that there is no universal formula that works for all quintics. A young French teenager, Évariste Galois took Abel's theorem a whole lot further. He was able to determine, in principle, which polynomials of any degree are soluble by radicals. In doing so he developed several whole new branches of mathematics.

It's hard to appreciate the enormous achievement of Galois. The Eiffel tower in Paris was built by Gustave Eiffel. Now imagine if he'd built the tower in the Middle Ages! He would have had to invent the art of steel-making as well as deloping the theory of structures and a whole lot of other technologies that were necessary to build the tower. Galois had to do that. The techniques he used couldn't be found in the text-books of his day. He had to build it all himself. In fact two of the courses we teach in third year at Macquarie owe their existence to that young teenager!

I'm often asked who are the greatest mathematicians of all time. My three, without doubt are Euclid, Newton and Galois. "What, not Einstein?" I'm often asked. There's a great misunderstanding about Albert Einstein. He wasn't a mathematician at all. He was well equipped with a deep knowledge of mathematics, but no more than many of his time. So far as I know he never added a single thing to mathematical knowledge.

On the other hand, if I was asked who were the greatest theoretical physicists of all time I would certainly have included him in my top three. But it's known that he often consulted his mathematical colleagues when he got stuck with some obscure mathematical technique. He had enormous insight as a theoretical physicist but please don't call him a mathematician.

Euclid was influential, not because of Euclidean geometry, but because he introduced the concept of *proof* into mathematics. Before then it was just an experimental science.

Newton was influential because he invented Calculus. This is one of the greatest tools of mathematics. Though, if I was German, I would have substituted Leibnitz in place of Newton. They both invented Calculus, independently, at about the same time. If they hadn't done it someone else would certainly have done so because science, and especially astronomy, had got to a point where calculus was needed.

Galois was influential, not because of his somewhat unimportant theorem about the impossibility of solving a quintic by radicals, but because of the methods that he had to invent to prove this theorem. He invented Group Theory. This is about finite mathematical systems that arose in connection with permutations of the zeros of polynomials, and the symmetries of these zeros. (Remember I said that symmetry is important in discussing polynomials.)

Then, later mathematicians realised that the theory that Galois developed for his groups applied to permutations of anything. Some time later it was realised that virtually everything in Group Theory could be derived from the four basic axioms that can define a group. This axiomatic approach, which admittedly Galois didn't invent, has now revolutionised the whole of mathematics, and nearly every branch of mathematics these days begins with a set of axioms.

It's true that Euclid began his geometry with a list of axioms. But they were considered to be "self-evident truths" describing the real world. The axioms of group theory, and countless other branches of modern mathematics, are essentially part of definition of some abtract structure. Without Galois it may have taken a long time for mathematics to reach the level of abstraction it has today. And don't think of 'abstract' as a dirty word meaning abstruse, any not relevant to the real world. 'Abstract' means 'powerful' and abstract mathematics is extremely useful in the modern world.

Galois' story is fascinating. He was a drop-out at school because he found what he was being taught boring. He read on his own as a hobby, but his real passion was being a political radical. He was in and out of jail and did most of his mathematics there. Most of his ground-breaking work was done at the age of 19. I point this out to my third year students in the Galois Theory course. They are mostly 19.

He was killed in a dual at the age of 20 but his mathematical achievements were completely unknown until his rather obscure and bady written manuscript was discovered by chance some thirty years after his death. He is still unknown to all but a few mathematical specialists. But he should be up there with the greats. Most universities run two courses based on his work – Group Theory and Galois Theory. If you want to read more about him, look at the Appendix in my *Galois Theory* notes.

So, there's no formula for the quintic, or higher degree polynomials – it stops at quartics. But even if there was such a formula we wouldn't use it. I've shown how, with the help of the Extended Newton's Method, we can, solve polynomials of any degree to any desired degree of accuracy.

Let me now show how we can solve certain higher degree polynomials by simply noticing some special pattern in the coefficients.

Example 10:
Solve $63z^6 + 192z^5 + 240z^4 + 160z^3 + 60z^2 + 12z + 1 = 0$.
Solution: You can be forgiven for not noticing the pattern of the coefficients, But notice that, the binomial coefficients 1, 6, 15, 20, 15, 6, 1 divide these coefficients and, with the exception of 63 which is almost a power of 2, the powers of 2 from 32, 16, 8, 4, 2 also divide these coefficients. In fact, again with the exception of the coefficient 63, each coefficient is a binomial coefficient times a power of 2.

binomial	1	6		15	20	15	6	1
power of 2	64	32		16	8	4	2	1
product	64	192		240	160	60	12	1

Adding z^6 to both sides of our equation we get:

$$(2z + 1)^6 = z^6.$$

Hence $\left(\frac{2z + 1}{z}\right)^6 = 1$.

The 6th roots of unity are 1, α, α^2, -1, α^4, α^5 where $\alpha = \cos(\pi/3) + \mathrm{i}\sin(\pi/3) = \frac{1 + \sqrt{3}\mathrm{i}}{2}$.

If α^r is any one of these 6th roots of unity, $2z + 1 = \alpha^r z$ and so $z = \frac{1}{\alpha^r - 2}$.

The two real solutions are -1 and $-1/3$.

$\alpha - 2 = \frac{-3 + \sqrt{3}\mathrm{i}}{2}$ and so $z = \frac{2(-3 - \sqrt{3}\mathrm{i})}{12} = \frac{-3 - \sqrt{3}\mathrm{i}}{6}$.

$\alpha^2 - 2 = \frac{-5 + \sqrt{3}\mathrm{i}}{2}$ and so $z = \frac{2(-5 - \sqrt{3}\mathrm{i})}{28} = \frac{-5 - \sqrt{3}\mathrm{i}}{14}$.

The other two zeros will be the conjugates of the above two. So the 6 solutions are:

$$-1, -\frac{1}{3}, \frac{-3 - \sqrt{3}\mathrm{i}}{6}, \frac{-3 + \sqrt{3}\mathrm{i}}{6}, \frac{-5 - \sqrt{3}\mathrm{i}}{14}, \frac{-5 + \sqrt{3}\mathrm{i}}{14}.$$

There are systematic algorithms for systems of polynomial equations, analogous to the Gaussian reduction to echelon form for linear systems. The method is known as finding a **Gröbner Basis** for the system. A considerable amount of work has been done in the last decade on Gröbner Bases in the context of *Computational Algebra*. However it's too technical for us to consider here. We'll content ourselves with simple-minded *ad hoc* methods as in the above examples.

§10.6. Solving Two Polynomials In Two Variables

Suppose we have a system $\left.\begin{matrix} \mathrm{a}(x, y) = 0 \\ \mathrm{b}(x, y) = 0 \end{matrix}\right\}$ where a(x, y) and b(x,y) are functions (not just polynomials) from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$. In my notes on *Galois Theory* I describe my Extended Newton's Method that handles the problem of solving such a system.

The ordinary Newton's Method, in effect finds the equation of the tangent at a certain point and works out where this tangent cuts the *x*-axis. Hopefully this is a better approximation.

The two functions in the system describe two surfaces in $\mathbb{R}^3$. If we have an approximation (x_0, y_0) to a solution for the system, the Extended Newton's Method, in effect, finds the equation of the tangent plane to each of these surfaces at $\mathrm{a}(x_0, y_0)$ and $\mathrm{b}(x_0, y_o)$. Each of these tangent planes cuts the *x*-*y* plane in a line and the Extended Method finds the point (x_1, y_1) of intersection of these lines. Hopefully this is a better approximation to the solution. This then becomes the new (x_0, y_0) and the process continues until, with a certain amount of luck, these approximations converge to a sufficiently good approximation.

For now I will describe another method that can be used when a(x, y) and b(x, y) are real polynomials that are symmetric in x, y.

To solve the system: $\left.\begin{array}{r} a(x, y) = 0 \\ b(x, y) = 0 \end{array}\right\}$ where a(x, y) and b(x, y) are symmetric in x and y:

(1) Transform the system by making the substitution:

$$S = x + y,\ P = xy.$$

(2) Solve for S and P

(3) Obtain x, y as the zeros of the quadratic

$$x^2 - Sx + P = 0.$$

Example 11: Solve the system:

$$\left.\begin{array}{r} x^3 + y^3 = 279 \\ x^2 + y^2 = 65 \end{array}\right\}$$

Solution: Let $S = x + y$ and $P = xy$.

Then $x^3 + y^3 = (x + y)^3 - 3x^2y - 3xy^2 = S^3 - 3PS$ and $x^2 + y^2 = S^2 - 2P$. This gives us a new system:

$$\left.\begin{array}{r} S^3 - 3PS = 279 \\ S^2 - 2P = 65 \end{array}\right\}$$

What have we gained by doing this? In terms of S we have a quadratic and a cubic, as before. But in terms of P we have two linear polynomials. So $P = \dfrac{S^3 - 279}{3S} = \dfrac{S^2 - 65}{2}$ (dividing by S is OK because S = 0 is not possible — why not?).

Thus $2S^3 - 558 = 3S^3 - 195S$ and so $S^3 - 195S + 558 = 0$. There are three real solutions to this cubic which we can find by Newton's Method: S = 3 and approximately
S = –15.22 and 12.22.
It's now easy to find the corresponding values of P.
The solutions for S and P are (approximately):

$$S = -15.22,\ P = 83.32;$$
$$S = 3,\ P = -28;$$
$$S = 12.22,\ P = 42.16$$

Now, in each of these cases, we must solve the system:

$$\left.\begin{aligned} x + y &= S \\ xy &= P \end{aligned}\right\}$$

The values of x, y will simply be the solutions of the quadratic equations $x^2 - Sx + P = 0$.

Solving these three quadratics we get six solutions for x and y, approximately:
(–4, 7), (7, –4),
(–7.61 +10.08 i , –7.61 –10.08 i),
(–7.61 –10.08 i , –7.61 +10.08 i),
(6.11 + 4.39 i , 6.11 – 4.39 i),
(6.11 – 4.39 i , 6.11 + 4.39 i).

EXERCISES FOR CHAPTER 10

Exercise 1: Solve $3x^2 - 4x + 3 = 0$.

Exercise 2: Solve the quadratic $z^2 + 2iz - (2 + \sqrt{3}i) = 0$.

Exercise 3: Solve the quadratic $z^2 + z - i = 0$. (Work to 3 decimal places.)

Exercise 4: Solve the cubic $2x^3 - 11x^2 + 14x - 3$. Work to 3 decimal places.

Exercise 5: Solve the cubic $2x^3 - 11x^2 + 14x - 3$ exactly, expressing the solutions as surds. But do not attempt to use the cubic formula!

Exercise 6: Solve the cubic $z^3 - 2z - 8 = 0$.

Exercise 7: Solve the cubic $z^3 - 7z - 6 = 0$.

Exercise 8: Solve $z^3 + z + i = 0$.

Exercise 9: Prove that if $z = x + iy$ is a non-real zero of a monic real cubic $f(z)$, then x is a real zero of the cubic

$$2f(x) - f'(x).f''(x).$$

Exercise 10: Solve $x^4 + 3x^2 + 4 = 0$ exactly, expressing the solutions in the form $x + iy$ where x, y are in surd form.

Exercise 11: Solve the quintic $3x^5 - 5x^3 + 1 = 0$.

Exercise 12: Solve the equation
$8x^7 + 28x^6 + 56x^5 + 70x^4 + 56x^3 + 28x^2 + 8x + 1 = 0$.

Exercise 13: Solve the system:

$$\left.\begin{array}{r} x^3y + xy^3 = 14 \\ x^2 + y^2 - xy = 5 \end{array}\right\}$$

Exercise 14: Solve the system:

$$\left.\begin{array}{r} x^2 + y^2 + x + y = 12 \\ x^2y + xy^2 + 2x + 2y = -2 \end{array}\right\}$$

SOLUTIONS FOR CHAPTER 10

Exercise 1: $\mathrm{x} = \dfrac{4 \pm \sqrt{16 - 36}}{6} = \dfrac{4 \pm \sqrt{-20}}{6} = \dfrac{2 \pm \sqrt{5}\,\mathrm{i}}{3}$.

Exercise 2: $\Delta = (2\mathrm{i})^2 + 8 + 4\sqrt{3}\mathrm{i}$

$$= 4 + 4\sqrt{3}\mathrm{i} = 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\right) = 2\mathrm{cis}\left(\frac{\pi}{3}\right).$$

$$\sqrt{\Delta} = \pm\sqrt{2}\mathrm{cis}\left(\frac{\pi}{6}\right) = \pm\sqrt{2}\left(\frac{\sqrt{3}}{2} + \frac{1}{2}\,\mathrm{i}\right) = \pm\left(\frac{\sqrt{3}}{\sqrt{2}} + \frac{1}{\sqrt{2}}\,\mathrm{i}\right).$$

Hence $\mathrm{z} = \dfrac{-2\sqrt{2}\,\mathrm{i} \pm (\sqrt{3} + \mathrm{i})}{2\sqrt{2}}$

$$= \frac{\sqrt{3}}{2\sqrt{2}} - \left(\frac{2\sqrt{2} - 1}{2\sqrt{2}}\right) \text{i and} - \frac{\sqrt{3}}{2\sqrt{2}} - \left(\frac{2\sqrt{2} + 1}{2\sqrt{2}}\right) \text{i.}$$

Exercise 3: $\Delta = 1 + 4\text{i}$. $|\Delta| = \sqrt{17}$.

arg z = θ where $\cos\theta = \frac{1}{\sqrt{17}}$ and $\sin\theta = \frac{4}{\sqrt{17}}$.

Hence θ = 1.326.

Thus $\sqrt{\Delta} = \pm 17^{1/4}$ (cos(θ/2 + i sin θ/2)

= ± 2.030(0.788 + 0.615)

= ± (1.600 + 1.248 i)

Hence $z = \frac{-1 + 1.600 + 1.248\text{ i}}{2}$ and $\frac{-1 - 1.600 - 1.248\text{ i}}{2}$

= 0.300 + 0.624 i and – 1.300 – 0.624 i.

Exercise 4: Using Newton’s Method we can locate 3 real solutions: 1.500, 3.732, 0.268.

Exercise 5: Using Newton’s Method we locate one real zero at 1.500. Checking we see that $x = 3/2$ is exactly one of the zeros. We can therefore divide the cubic by $2x - 3$ to obtain the quadratic $x^2 - 4x + 1 = 0$. Solving by the quadratic formula we can find the three solutions exactly as 3/2, 2 + √3 and 2 – √3.

Exercise 6: Put $z = x + \text{i}y$. Then

$$(x+\text{i}y)^3 - 2(x+\text{i}y) - 8 = 0.$$

Equating real and imaginary parts we arrive at the system:

$$\left.\begin{aligned} x^3 - 3xy^2 - 2x - 8 = 0 \\ 3x^2y - y^3 - 2y = 0 \end{aligned}\right\}$$

If $y \neq 0$ the last equation reduces to:

$$3x^2 - y^2 - 2 = 0 \text{ and so } y^2 = 3x^2 - 2.$$

Substituting into the first equation we get:

$$x^3 - 3x(3x^2 - 2) - 2x - 8 = 0.$$

Simplifying:

$8x^3 - 4x + 8 = 0$ and so $f(x) = 2x^3 - x + 2 = 0$.

This has a real root close to $x = -1$.

Now $f'(x) = 6x^2 - 1$ and by Newton's Method we get a a better approximation:

$$-1 - f(-1)/f'(-1) = -1 - 1/5 = -1.2$$

An even better approximation is:

$$-1.2 - f(-1.2)/f'(-1.2) = -1.2 + \frac{0.2567}{7.64} = -1.166.$$

The corresponding values of y are $\pm\sqrt{3(-1.166)^2 - 2} = \pm\sqrt{2.079} = \pm 1.442$.

There are thus two non-real roots: -1.166 ± 1.442 i.

Since the sum of the roots is 0, the real zero must be approximately 4.332.

Exercise 7: Putting $z = x + iy$ as above we get the system:

$$\left.\begin{aligned} x^3 - 3xy^2 - 7x - 6 = 0 \\ 3x^2y - y^3 - 7y = 0 \end{aligned}\right\}$$

Dividing the second equation by y we get $y^2 = 3x^2 - 7$.

Substituting into the first equation, and simplifying, we get $4x^3 - 7x + 3 = 0$.

Clearly $x = 1$ is one root and factorising we get $4x^3 - 7x + 3 = (x - 1)(4x^2 + 4x - 3)$

$= (x - 1)(2x - 1)(2x + 3)$.
So $x = 1$, 1/2 and –3/2. The values of y^2, from $y^2 = 3x^2 - 7$ are –4, –6.25, –4.75.

But wait, there appears to be something wrong here! The values of x and y have to be real so y^2 can't be negative. Does this mean that there are no roots? No. Every cubic has three roots over the complex field. Moreover, one of them must be real.

What has happened is that in dividing by y we were implicitly assuming that $y \neq 0$, that is, we were only going in search of non-real roots. As it was we showed that there were none. All three roots must be real. This can be confirmed by sketching the original cubic. In fact it has roots at z = –1, –2 and 3.

Exercise 8: Let z = $x + iy$.
Then $(x + iy)^3 + (x + iy) + i = 0$.
Equating real and imaginary parts we get the system:

$$\left.\begin{aligned} x^3 - 3xy^2 + x &= 0 \\ 3x^2y - y^3 + y + 1 &= 0 \end{aligned}\right\}$$

If $x = 0$ then $y^3 - y - 1 = 0$.
By Newton's Method we get $y \approx 1.3247$.
Now suppose that $x \neq 0$.

Then, from the first equation, $x^2 - 3y^2 + 1 = 0$, and so $x^2 = 3y^2 - 1$.

Substituting into the second equation we get $3(3y^2 - 1)y - y^3 + y + 1 = 0$, that is

$$8y^3 - 2y + 1 = 0.$$

By Newton's Method we get $y \approx -0.6624$. Hence $x^2 = 0.3163$ and so $x = \pm\, 0.5624$.

Hence the three solutions to $z^3 + z + i = 0$ are, approximately:

$1.3247i$, $0.5624 - 0.6624i$ and $-0.5624 - 0.6624i$.

Exercise 9: Let $f(z) = z^3 + az^2 + bz + c$, where a, b, c are real, and let $z = x + iy$.

Then $(x + iy)^3 + a(x + iy)^2 + b(x + iy) + c = 0$.

Equating real and imaginary parts we get the system:

$$\left.\begin{array}{c} x^3 - 3xy^2 + ax^2 - ay^2 + bx + c = 0 \\ 3x^2y - y^3 + 2axy + by = 0 \end{array}\right\}$$

Since z is non-real, $y \neq 0$ and so $3x^2 - y^2 + 2ax + b = 0$.

So $y^2 = 3x^2 + 2ax + b$.

Substituting into the first equation we get:

$(x^3 + ax^2 + bx + c) - (3x + a)(3x^2 + 2ax + b) = 0$ and so $2(x^3 + ax^2 + bx + c) - (6x + 2a)(3x^2 + 2ax + b) = 0$. In other words: $2f(x) - f''(x).f'(x) = 0$.

Exercise 10: The obvious way to solve this is to treat it as a quadratic in x^2.

So $x^2 = \dfrac{-3 \pm \sqrt{9 - 16}}{2} = \dfrac{-3 \pm \sqrt{7}\, i}{2}$.

But now have to find the square roots of these numbers. This doesn't look too promising.

Let's try the technique given in the notes.

Let $x^4 + 3x^2 + 4 = (x^2 + ax + \mathrm{b})(x^2 + cx + \mathrm{d})$ where a, b, c, d are real.

Then $$\left.\begin{array}{c} a + c = 0 \\ ac + b + d = 3 \\ ad + bc = 0 \\ bd = 4 \end{array}\right\}.$$

Hence $c = -a$ and $d = 4/b$.

Substituting into the middle two equations we get:

$$\left.\begin{array}{r} -a^2 + b + \dfrac{4}{b} = 3 \\ \dfrac{4a}{b} - ab = 0 \end{array}\right\}.$$

Simplifying:

$$\left.\begin{array}{r} -a^2b + b^2 + 4 = 3b \\ a(4 - b^2) = 0 \end{array}\right\}$$

So $a = 0$ or $b = \pm 2$.

If $a = 0$, $b^2 - 3b + 4 = 0$ and the solutions for b are not real.

If $b = -2$, $a^2 = -7$ and again a is not real.

If $b = 2$ and $a = 1$. Then $c = -1$ and $d = 2$.

This gives the factorisation $(x^2 + x + 2)(x^2 - x + 2)$.

If $b = 2$ and $a = -1$. Then $c = 1$ and $d = 2$. This gives the factorisation $(x^2 - x + 2)(x^2 + x + 2)$, which is the same.

So the solutions are : $x = \dfrac{\pm 1 \pm \sqrt{7}\mathrm{i}}{2}$ (all 4 combinations).

Exercise 11: This is a quintic which is not soluble by radicals. That is, there is no exact surd expression for the solutions. (See my notes on *Galois Theory* where this is proved.) But we it has at least one real solution so we set up our Newton's Method spreadsheet. We find, in fact, that there are three real zeros:
$\alpha = 1.217$, $\beta = 0.643$ and $\gamma = -1.343$ (to 3 decimal places).
So $(x - \alpha)(x - \beta)(x - \gamma)$ divides $3x^5 - 5x^3 + 1$.
Let $3(x - \alpha)(x - \beta)(x - \gamma)(x^2 + ax + b) = 3x^5 - 5x^3 + 1$.
Then, equating the coefficients of x^4,
$3(\alpha + \beta + \gamma) + 3a = 0$, that is $a = -(\alpha + \beta + \gamma)$
and equating the constant terms, $-3\alpha\beta\gamma b = 1$, that is,
$b = -\dfrac{1}{\alpha\beta\gamma}$.
Now $\alpha + \beta + \gamma = 0.517$ and $\alpha\beta\gamma = -0.951$ so
$a = -0.517$ and $b = 1.052$.
Solving $x^2 - 0.517x + 1.052$ we get $x = 0.259 \pm 0.993$ i.
So the solutions are:
1.217, 0.643, -1.343 and 0.259 ± 0.993 i.

Exercise 12: Before you start reaching for Newton's Method, did you notice that the coefficients are mostly binomial coefficients.
In fact we can write the equation as $(x + 1)^8 = x^8$.
Let $Z = \dfrac{x+1}{x}$. Then $Z^8 = 1$.
The solutions for Z are $\text{cis}(2\pi k/8)$ for $k = 0, 1, \ldots, 7$.

Now $\text{cis}(\pi/4) = \dfrac{1 + i}{\sqrt{2}}$ and $\text{cis}(\pi/2) = i$ and $\text{cis}(\pi) = -1$. From these we can generate the 8^{th} roots of unity as: 1, -1, $\pm i$, and $\dfrac{\pm 1 \pm i}{\sqrt{2}}$.

Now $x = \dfrac{1}{Z - 1}$. This eliminates Z = 1. We knew this would happen because we have 8 possibilities but our equation will have only 7 solutions.

$$-\frac{1}{2}, -\frac{1 \pm i}{2}, -\frac{1}{2} \pm \left(\frac{\pm\sqrt{2} + 1}{12}\right) i.$$

Exercise 13: This system is symmetric in x and y. Therefore put $S = x + y$ and $P = xy$.

The system reduces to:

$$\left.\begin{aligned} P(S^2 - 2P) &= 14 \\ S^2 - 3P &= 5 \end{aligned}\right\}$$

Noting the similarity of the $S^2 - 2P$ with the $S^2 - 3P$ we can write the first equation as:

$$P(P + 5) = 14.$$

Hence $P^2 + 5P - 14 = (P - 2)(P + 7) = 0$ and so

$$P = 2 \text{ or } -7.$$

If $P = 2$, $S^2 = 3P + 5 = 11$ and so $S = \pm\sqrt{11}$.

This leads to the two quadratics in x, y:

$x^2 \pm\sqrt{11}\, x + 2$ giving the solutions:

$x = \dfrac{\sqrt{11} + \sqrt{3}}{2}$, $y = \dfrac{\sqrt{11} - \sqrt{3}}{2}$ (or vice versa) and $x = \dfrac{-\sqrt{11} + \sqrt{3}}{2}$, $y = \dfrac{-\sqrt{11} - \sqrt{3}}{2}$ (or vice versa).

If P = –7, S = ±4i giving the quadratics:

$x^2 \pm 4i\,x - 7$ and hence the solutions:

$x = \sqrt{3} + 2i$, $y = -\sqrt{3} + 2i$ (or vice versa) and $x = \sqrt{3} - 2i$, $y = -\sqrt{3} - 2i$ (or vice versa).

Exercise 14:

This system is symmetric in *x*, *y*. Let S = *x* + *y* and P = *xy*.

Then $S^2 - 2P + S = 12$ and $PS + 2S = -2$.

From the first equation $P = \frac{1}{2}(S^2 + S - 12)$.

Substituting into the second we get $S^3 + S^2 - 8S + 4 = 0$.

By inspection, S = 2 is a solution. Hence S – 2 is a factor of this cubic.

The other factor can be easily found to be $S^2 + 3S - 2$, giving $S = \dfrac{-3 \pm \sqrt{17}}{2}$.

The corresponding values of P are –3, – 7/2, – 7/2.

The solutions *x*, *y* are the two zeros of the three quadratics $x^2 - Sx + P$.

So $x, y = \dfrac{S \pm \sqrt{S^2 - 4P}}{2}$.

If S = 2, P = – 3, x, y = $\dfrac{2 \pm \sqrt{4 + 12}}{2}$, that is, $x = 3$, $y = -1$ or $x = -1$, $y = 3$.

If $S = \frac{-3 + \sqrt{17}}{2} \approx 0.56155$ and $P = -3.5$ then x, y = 2.17256, –1.61100.

If $S = \frac{-3 - \sqrt{17}}{2} \approx 3.56155$ and $P = -3.5$ then x, y = 8.02083, –4.36363.

So there are 6 solutions:

x	–1	3	2.1726	–1.6110	8.0208	–4.3636
y	3	–1	–1.6110	2.1726	–4.3636	8.0208